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HW4

PHYS 515B
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1. Express the acceleration $\dot{\mathbf{v}}$ of a particle in terms of its velocity and electromagnetic fields $\mathbf{E}$, $\mathbf{B}$.

   Solution. Start with

$$\begin{aligned}\frac{d\vec{p}}{dt} &= q\vec{E} + \frac{q}{c}(\vec{v}\times\vec{B}) \\ \vec{p} &= \frac{m}{\sqrt{1-v^2/c^2}}\vec{v} = \frac{\Sigma}{c^2}\vec{v} \\ \vec{v} &= \frac{d\vec{x}}{dt} \\ \Sigma &= \frac{mc^2}{\sqrt{1-v^2/c^2}}.\end{aligned}$$

   It is easy to check $\frac{d\Sigma}{dt} = \vec{v}\cdot\frac{d\vec{p}}{dt} = q\vec{E}\cdot\vec{v}$. Now the acceleration $\dot{\vec{v}}$ is

$$\dot{\vec{v}} = \frac{q}{m}\sqrt{1-v^2/c^2}\big[\vec{E} + \frac{1}{c}(\vec{v}\times B) - \frac{1}{c^2}\vec{v}(\vec{v}\cdot\vec{E})\big].$$

2. A particle of mass m and charge q is accelerated by a uniform electric field E from time $t=0$ to T. The initial velocity of the particle is zero, find the velocity of the particle at T.

   Solution. Assume electric field is in x-direction. Particle is initially at rest, so $p_x = p_y = p_z = 0$. Additionally, $\dot{p}_y = \dot{p}_z = 0$. For the x-direction, $\dot{p}_x = qE$. So, the momentum integrates to

$$p_x = \int \dot{p}_x dt = qEt + C = qEt \quad \text{by initial conditions.}$$

   Since particle is initially at rest, $\mathrm{E}_0 = mc^2$. The energy of the particle evolves as

$$\begin{aligned}\mathrm{E} &= \sqrt{m^2c^4 + c^2(qEt)^2} \\ &= \sqrt{\mathrm{E}_0^2 + c^2(qEt)^2}.\end{aligned}$$

   Take relativistic momentum into account

$$\begin{aligned}p_x = mc\frac{dU}{ds} &= \frac{1}{c^2}\left(\frac{mc^2}{\sqrt{1-v^2/c^2}}\right)\frac{dx}{dt} \\ \Longrightarrow \quad \left.\frac{dx}{dt}\right|_{t=T} = v(T) &= \frac{c^2 p_x}{\mathrm{E}_0} \\ &= \frac{c^2 qET}{\sqrt{m^2c^4 + (cqET)^2}}.\end{aligned}$$

3. Determine the frequency of vibration of a charged oscillator placed in a constant, uniform magnetic field. The proper frequency of vibration of the oscillator (in the absence of the field) is ω_0.

   Solution. In the absence of a magnetic field, the dynamical equation is

$$\ddot{\vec{r}} + \omega_0^2 \vec{r} = 0.$$

   If the magnetic field is in the z-direction, the equation above becomes

$$\begin{aligned} \ddot{x} + \omega_0^2 x &= \frac{qB}{mc}\dot{y} \\ \ddot{y} + \omega_0^2 y &= -\frac{qB}{mc}\dot{x} \\ \ddot{z} + \omega_0^2 z &= 0. \end{aligned}$$

   Combine the first two equations into one equation with $\zeta = x + iy$,

$$\ddot{\zeta} + \omega_0^2 \zeta = -i\frac{qB}{mc}\dot{\zeta}.$$

   The frequency of vibration in the $x - y$ plane is

$$\omega = \sqrt{\omega_0^2 + \frac{1}{4}\left(\frac{qB}{mc}\right)^2} \pm \frac{qB}{2mc}.$$

4. Determine the relativistic motion of a charge q in parallel uniform electric and magnetic fields.

   Solution. The magnetic field has no influence on the motion along the direction of $\vec{E}$ and $\vec{B}$ (in z-direction). So,

$$z = \frac{\Sigma}{qE}, \quad \Sigma = \sqrt{\Sigma_0^2 + (cqEt)^2}.$$

   For the motion in the $x - y$ plane, the equations are

$$\begin{aligned} \dot{p}_x &= \frac{q}{c}Bv_y \\ \dot{p}_y &= -\frac{q}{c}Bv_x. \end{aligned}$$

   Combine into one equation

$$\begin{aligned} \frac{d}{dt}(p_x + ip_y) &= -i\frac{qB}{c}(v_x + iv_y) \\ &= -i\frac{qBc}{\Sigma}(p_x + ip_y). \end{aligned}$$

   The solution of this equation is

$$p_x + ip_y = p_t e^{-i\phi}$$

where p_t is a constant, ϕ is given by

$$\begin{aligned} d\phi &= \frac{qBc}{\Sigma} dt \\ \implies ct &= \frac{\Sigma_0}{qE} \sinh \frac{E}{B} \phi \end{aligned}$$

Now, use the fact that

$$p_x = \frac{\Sigma}{c^2} \dot{x}, \ p_y = \frac{\Sigma}{c^2} \dot{y}.$$

to get

$$\begin{aligned} p_t e^{-\phi} &= \frac{\Sigma}{c^2} (\dot{x} + i\dot{y}) \\ &= \frac{\Sigma}{c^2} \frac{d\phi}{dt} \left(\frac{dx}{d\phi} + i \frac{dy}{d\phi} \right) \\ &= \frac{qB}{c} \frac{d}{d\phi} (x + iy). \end{aligned}$$

The solution of which becomes

$$\begin{aligned} x &= \frac{cp_t}{qB} \sin \phi \\ y &= \frac{cp_t}{qB} \cos \phi \\ z &= \frac{\Sigma_0}{qE} \cosh \frac{E}{B} \phi. \end{aligned}$$

5. (a) Two charge q particles 1 and 2 with distance d in z-direction of rest frame, find the acceleration of particle 1 due to the electric field of particle 2.

   Solution. Consider the particles are initially at rest. The only direction of interest in this problem is the z-direction.

   Since we are considering the rest frame, consider the non-relativistic momentum $\dot{p}_z = m\dot{v}_z$. The electric force experienced at particle 1 as a result of particle 2 is

$$\begin{aligned} m\dot{v}_z &= \frac{q^2}{d^2} \\ \implies \dot{v}_z &= \frac{q^2}{md^2} \end{aligned}$$

   (b) If both particles move in z-direction with high speed $\gamma = 1/\sqrt{1 - v^2/c^2}$, find the acceleration of particle 1 due to the electromagnetic field of particle 2.

   Solution. In frame, in which both particles move with high speed $\gamma = 1/\sqrt{1 - v^2/c^2}$ in z-direction. Since electric and magnetic fields in the z-direction

$$E_\parallel = E'_\parallel, \quad B_\parallel = B'_\parallel.$$

The force on the particle 2 on particle 1 is

$$\hat{z} \cdot \vec{F} = q\vec{E} \cdot \hat{z} = q\vec{E}' \cdot \hat{z} = \hat{z} \cdot \vec{F}'.$$

From the solution of problem (1), with $\dot{\vec{v}}$ is in the z-direction.

$$\begin{aligned}
\dot{\vec{v}} &= \frac{q}{m}\sqrt{1 - v^2/c^2}\big[\vec{E} + \frac{1}{c}(\vec{v} \times B) - \frac{1}{c^2}\vec{v}(\vec{v} \cdot \vec{E})\big] \\
&= \frac{q}{m}\left(1 - \frac{v^2}{c^2}\right)^{3/2} \vec{E} \cdot \hat{z} \\
&= \left(1 - \frac{v^2}{c^2}\right)^{3/2} \vec{a}'.
\end{aligned}$$

6. The Lagrangian of a relativistic charged particle of mass m and charge q is

$$L = -mc^2\sqrt{1 - v^2/c^2} + \frac{q}{c}\mathbf{A} \cdot \mathbf{v}.$$

Assuming magnetic field is given by a magnetic dipole p_m along the polar axis, the vector potential is $\mathbf{A} = \frac{p_m \sin\theta}{r^2}\mathbf{e}_\phi$ where θ is the polar angle, and ϕ the azimuthal angle.

(a) Express the canonical momentum p_ϕ conjugate to ϕ in their coordinates and their derivatives.

Solution. Since the force is purely magnetic, if I consider that the initial velocities, $\dot{r}, \dot{\theta} = 0$, and the initial positions $r, \theta = C$, then $r, \theta = C$ for all time. The Lagrangian can be written as

$$L = -mc^2\sqrt{1 - \dot{\phi}^2/c^2} + \frac{p_m \sin\theta}{r^2}\dot{\phi}.$$

The momentum can be written as

$$\begin{aligned}
p_\phi &= \frac{\partial L}{\partial \dot{\phi}} \\
&= -\frac{\dot{\phi} mc^2}{\sqrt{1 - \dot{\phi}^2/c^2}} + \frac{p_m \sin\theta}{r^2}.
\end{aligned}$$

Use the following relationships

$$\begin{aligned}
E &= \frac{mc^2}{\sqrt{1 - \dot{\phi}^2/c^2}} \\
\dot{\phi} &= \frac{c^2 p_\phi}{E}.
\end{aligned}$$

The result becomes

$$p_\phi = \frac{p_m \sin\theta}{2r^2}.$$

(b) Show that the momentum p_ϕ is a constant of the motion.

Solution. Since the derivatives $\dot{r}, \dot{\theta} = 0$,

$$\begin{aligned} \dot{p}_\phi &= \frac{d}{dt}\left(\frac{p_m \sin\theta}{2r^2}\right) \\ &= 0 \end{aligned}$$

which shows that $p_\phi = C$, some constant, for all time. So, p_ϕ is a constant of the motion.